AN APPROXIMATE SOLUTION OF THE PROBLEM INVOLVED IN THE HEATING OF STEEL WITH MINIMUM DECARBONIZATION

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Inzhenerno-Fizicheskii Zhurnal, Vol. 15, No. 2, pp. 280–285, 1968

UDC 669.046

The problem of optimum control for the process of metal heating is formulated so as to achieve minimum decarbonization. The calculational formulas are presented for the local numerical method of an approximate solution for the problem.

The problem of selecting the temperature regime for a furnace so as to ensure the heating of a blank to the required specifications with minimum decarbonization of the surface can be rigorously formulated within the terms of the theory of optimum control of systems with distributed parameters [1]. Problems of this type were initially associated with minimizing the amount of scale during heating; solutions were formulated in [2] on the assumption that the process under consideration is described by a system of 2-nd- or 3-rd-order ordinary differential equations.

An attempt is made below to approach the solution of the optimum problem when the controlled object is essentially described by nonlinear heat-conduction equations and the corresponding equation of diffusion for carbon [3]. For the case of the heating of plates, we derive the calculation formulas for the local numerical method proposed in [4] for the solution of the problem.

Let the equations for the heating [5] of a plate have the form

$$\xi(Q)\frac{\partial Q}{\partial t}=\frac{\partial}{\partial x}\left(\lambda(Q)\frac{\partial Q}{\partial x}\right),\quad 0\leqslant x\leqslant 1,\ 0\leqslant t\leqslant T; \tag{1}$$

$$Q(x,\ 0)=Q_0(x);\ \left.\frac{\partial Q}{\partial x}\right|_{x=1}=0;$$

$$\left.\frac{\partial Q}{\partial x}\right|_{x=0}=\alpha(R^4-Q_s^4); \tag{2}$$

$$\nu\frac{dR(t)}{dt}+R(t)=u(t),\ R(0)=R_0 \tag{3}$$

and at the instant at which the heating process T is concluded we have to have the specified mean square deviation from the plate distribution Q*(x) required from the standpoint of the equipment, i.e., the following relationship must be satisfied:

$$\text{const}=A=I_1=\int_0^1[Q(x,\ T)-Q^*(x)]^2\,dx. \tag{4}$$

The control function u(t) is subject to the limitation

$$0\leqslant u(t)\leqslant 1. \tag{5}$$

Let us describe the decarbonization process for the surface [6] with the equations

$$\frac{\partial C}{\partial t}=D(Q_s)\frac{\partial^2 C}{\partial x^2},\quad 0\leqslant x\leqslant 1;\ 0\leqslant t\leqslant T, \tag{6}$$

$$C(x,\ 0)=C_0,\ \left.\frac{\partial C}{\partial x}\right|_{x=1}=0,\ C(0,\ t)=C_s \tag{7}$$

where $D(Q_S)$ is the coefficient of carbon diffusion determined, for example, from the empirical Welles and Mell formula [6]

$$D(Q_s)=\{0.07+0.06[C_0]\}\exp\left(-\frac{3200}{1.99[Q_s+273]}\right),$$

and we will evaluate the thickness of the decarbonized layer at the instant T by means of the functional

$$I_0=\int_0^1[C_0-C(x,\ T)]\,dx. \tag{8*}$$

Obviously, $I_0 = 0$ when there is no decarbonization.

Let us solve the following optimum problem: for system (1)–(3), (6)–(7), select the control function u(t), $0 \leq t \leq T$ (the time T is fixed), constrained by condition (5) and such that the functional I_0, given by formula (8*), assumes its minimum possible value, while the functional I_1, given by formula (4), assumes the specified value at the instant $t = T$. The solution of the formulated problem is attained by the method of successive approximations, based on the construction of a transition from the control function $u^k(t)$ (achieved at the k-th iteration) to the function $u^{k+1}(t)$ near to it, such that the value of I_0 diminishes, i.e., $I_0(u^k(t)) \geq I_0(u^{k+1}(t))$, while the value of I_1 remains constant to some degree of accuracy. To construct such a process let us determine the relationship between the variations of the functionals δI_0 and δI_1 and the small variation in the control $\delta u(t)$. Using the results from perturbation theory [7], and carrying out the appropriate calculations, we derive formulas for this relationship in the form

$$\delta I_0=\int_0^T\tilde{\psi}_0(t)\,\delta u(t)\,dt, \tag{8}$$

$$\delta I_1=\int_0^T\psi_0(t)\,\delta u(t)\,dt. \tag{9}$$

Here $\tilde{\psi}_0(t)$ satisfies the following system of equations:

$$\frac{\partial\psi}{\partial t}+D(Q_s)\frac{\partial^2\psi}{\partial x^2}=0,\ 0\leqslant x\leqslant 1,\ 0\leqslant t\leqslant T, \tag{10}$$

$$\psi(x,\ T)=-1,\ \left.\frac{\partial\psi}{\partial x}\right|_{x=1}=0,\ \psi(0,\ t)=0, \tag{11}$$

$$\xi(Q)\frac{\partial\tilde{\psi}}{\partial t}+\frac{\partial}{\partial x}\left(\lambda(Q)\frac{\partial\tilde{\psi}}{\partial x}\right)-\frac{\partial\tilde{\psi}}{\partial x}\frac{\partial\lambda(Q)}{\partial x}=0, \quad (12)$$

$$\nu\frac{d\tilde{\psi}_0}{dt}-\tilde{\psi}_0(t)-4\alpha R^3\tilde{\psi}(0,\ t)=0, \quad (13)$$

$$\tilde{\psi}(x,\ T)=0,\ \tilde{\psi}_0(T)=0,\ \frac{\partial\tilde{\psi}}{\partial x}\Big|_{x=1}=0, \quad (14)$$

$$\left[\lambda(Q_s)\frac{\partial\tilde{\psi}}{\partial x}-4\alpha Q_s^3\tilde{\psi}\right]\Big|_{x=0}$$

$$\left\{\int_0^1\psi(x,\ t)\frac{\partial^2 C}{\partial x^2}\,dx\right\}\frac{\partial D}{\partial Q_s}. \quad (15)$$

In relationship (15) $\psi(x, t)$ is determined from the solution of system (10)–(11).

The function $\psi_0(t)$ satisfies the equations

$$\xi(Q)\frac{\partial\psi'}{\partial t}+\frac{\partial}{\partial x}\left(\lambda(Q)\frac{\partial\psi'}{\partial x}\right)-\frac{\partial\psi'}{\partial x}\frac{\partial\lambda(Q)}{\partial x}=0, \quad (16)$$

$$\nu\frac{d\psi_0}{dt}+\psi_0-4\alpha R^3\psi'(0,\ t)=0,$$

$$0\leqslant t\leqslant T,\ 0\leqslant x\leqslant 1, \quad (17)$$

$$\psi_0(T)=0,$$

$$\psi'(x,\ T)=\frac{2}{\xi(Q(x,\ T))}\left[Q(x,\ T)-Q^*(x)\right], \quad (18)$$

$$\frac{\partial\psi'}{\partial x}\Big|_{x=1}=0,\ \left[\lambda(Q_s)\frac{\partial\psi'}{\partial x}+4\alpha Q_s^3\psi'\right]\Big|_{x=0}=0. \quad (19)$$

By means of formulas (8) and (9) we construct the process of successive approximations according to the usual scheme [4].

Let us divide the time axis into N parts with the points $0 \leq t_0 \leq t_1 \leq \ldots \leq t_N = T$ and we will deal only with the piecewise-constant controls of the form

$$u(t)=u_n \text{ where } t\in[t_{n-1},\ t_n]. \quad (20)$$

We will replace formulas (8) and (9) by finite-difference relationships of the form

$$\delta I_0^k=\sum_{n=1}^{N}\tilde{\psi}_{0n}^k\,\delta u_n^k, \text{ where } \tilde{\psi}_{0n}^k=\int_{t_{n-1}}^{t_n}\tilde{\psi}_0^k(t)\,dt, \quad (21)$$

$$\delta I_1^k=\sum_{n=1}^{N}\psi_{0n}^k\,\delta u_n^k, \text{ where } \psi_{0n}^k=\int_{t_{n-1}}^{t_n}\psi_0^k(t)\,dt, \quad (22)$$

where k is the iteration number; the numbers $\tilde{\psi}_{0n}^k$ and ψ_{0n}^k ($n = 1, 2, \ldots, N$) are partial derivatives of the functionals I_0^k and I_1^k with respect to the variables δu_n^k. The variation of δu_n^k ($n = 1, 2, \ldots, N$) is achieved by solving the following linear-programming problem.

Find N numbers δu_n^k minimizing the linear formula (21) and satisfying the following limitations:

$$\delta I_1^k=\sum_{n=1}^{N}\psi_{0n}^k\,\delta u_n^k=0, \quad (23)$$

$$0\leqslant u_n^k+\delta u_n^k\leqslant 1,\ |\delta u_n^k|\leqslant\varepsilon^k>0, \quad (24)$$

$$n=1,\ 2,\ \ldots,\ N.$$

The methods for the solution of such problems have now been well developed [8].

Finally, the scheme for the method of successive approximations is the following.

Let the control function $u^k(t)$ derived on the k-th iteration be specified. For the determination of $u^{k+1}(t)$ the following calculations are carried out.

1. The function $u^k(t)$ is substituted into (3), and (1)–(3) and (6)–(7) are integrated by one of the numerical methods. From (4)–(8) we calculate the values of I_0^k and I_1^k.

2. The functions $Q^k(x,t)$, $C^k(x,t)$, $R^k(t)$, $0 \leq t \leq T$, $0 \leq x \leq 1$ thus derived are substituted into (10)–(15) and (16)–(19), and these are then integrated in reverse order (the initial conditions in these equations are specified, as usual [4], at the instant $t = T$) as follows: initially system (10)–(11), then the derived function $\psi(x,t)$ is substituted into (15) and system (12)–(15) is integrated. Following this, system (16)–(19) is integrated.

3. We use formulas (21) and (22) to calculate the numbers $\tilde{\psi}_{0n}^k$ and ψ_{0n}^k ($n = 1, 2, \ldots, N$).

4. We select the magnitude of ε^k, and the variations in the control δu_n^k ($n = 1, 2, \ldots, N$) are determined by solution of the linear-programming problem.

5. The improved control is determined by the formula

$$u_n^{k+1}=u_n^k+\delta u_n^k,\ n=1,\ 2,\ \ldots,\ N. \quad (25)$$

Since formulas (8)–(9) and (21)–(22) are valid only when the first approximation is applicable (given sufficiently small $\delta u(t)$ and δu_n^k), the correct selection of ε^k at each iteration is a basic prerequisite for successful optimization. We can control the validity of the selection of ε_k by comparing the "true" (derived by iteration) increments of the functionals

$$\Delta I_1=I_1^{k+1}-I_1^k,\ \Delta I_0=I_0^{k+1}-I_0^k$$

and the increments δI_1 and δI_0 predicted on the basis of (21) and (22).

Remarks: 1) The storage capacity of a digital computer is governed by the condition that at each iteration it is necessary to store the distributions $Q(x,t)$, $R(t)$, $C(x,t)$, $0 \leq x \leq 1$, $0 \leq t \leq T$ at a sufficient number of points. In the linear case—when λ and ξ are constants—we need only the functions $Q_s(t)$, $R(t)$, and $C(x,t)$ for the solution of systems (10)–(15) and (16)–(19).

2) We should note that system (6)–(7) for carbon is integrated essentially in the narrow layer adjacent to the surface for $0 \leq x \leq x_0$, where $x_0 \ll 1$.

3) With limitations on such phase variables of the system as furnace temperature, plate surface, etc., we should proceed as recommended in [4].

4) The method is convenient for the compilation of standard programs to solve problems of this kind. In this case, the solutions of the analysis problem available because of the existence of a program (integration of system (1)–(3), (6) and (7) for a specified control function) make up a component part of the over-all algorithm scheme. Solutions of systems such as (10)–(15) and (16)–(19), as a rule, present no significant difficulties in view of their linearity.

5) Derivation of formulas such as (8) and (9) is possible for bodies of other shapes (cylinders, prisms)

and for another form of Eq. (3) which describes the furnace inertia. This equation, generally speaking, can be replaced by a nonlinear system of ordinary differential equations with specified initial conditions at the instant t = 0.

6) The problem of optimum control with other optimality criteria can be solved in similar fashion. For example, the problem of the fastest heating of a blank to a specified condition with a specified value for the magnitude of decarbonization at the end of the process, etc.

7) The mathematical model for the process of carbon diffusion in the surface layer of a metal can be changed by consideration of a boundary condition of the 3rd kind instead of condition (7), and also by introducing a second control function, i.e., the time-varying carbon potential of the atmosphere. This is essential, for example, in problems of carbonization.

In conclusion, let us take note that numerous important industrial processes (drying, carbonization of metal surfaces in special atmospheres, metal oxidation [9], etc.) have mathematical models similar to those considered above, and the problems of choosing the optimum production regime for these processes can be resolved on the basis of the methodology presented here. In this article we have not dealt with problems pertaining to the convergence of the method of successive approximations, nor is the rate of its convergence evaluated.

In [10], using specific numerical calculations, we have demonstrated the effectiveness of applying an analogous method of successive approximations to problems of optimum control in the heating of massive bodies, said method formulated in reference [5]. This circumstance may serve as a basis for the application of this method to the solution of problems similar to those treated above.

NOTATION

$Q(x,t)$ is the temperature distribution in a plate; $Q_S(t)$ is the temperature of the body surface; $R(t)$ is the temperature of the furnace; $u(t)$ is the thermal power of the furnace; $Q_0(x)$ and R_0 are the initial temperatures of the body and furnace; t is the time; x is the coordinate; T is the total heating time; $\lambda(Q)$ is the thermal conductivity of the material; $c(Q)$ and $\gamma(Q)$ are the heat capacity and density of the material; α and ν are the constants characterizing heat transfer in the "furnace–heated g body" system; $C(x,t)$ is the concentration of carbon at some distance from surface with given t, %; C_0 is the initial carbon content in steel, %; C_S is the carbon content on the surface of the metal, %; $\tilde{\psi}(x,t)$, $\psi(x,t)$, $\psi'(x,t)$, $\tilde{\psi}_0(t)$, and $\psi_0(t)$ are the functions satisfying systems of equations (10)–(15), (16)–(19); $n = 0, 1, \ldots, N$ are the numbers of the time intervals; k is the iteration number.

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12 October 1967

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